Non-Existence of Black Hole Solutions to Static, Spherically Symmetric Einstein-Dirac Systems – a Critical Discussion

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Abstract

This short note compares different methods to prove that Einstein-Dirac systems have no static, spherically symmetric solutions.

There are several approaches to prove that Einstein-Dirac systems do not admit static, spherically symmetric solutions. The first paper in this direction is [2], where it is shown that the Dirac equation has no normalizable, time-periodic solutions in the Reissner-Nordström geometry. In [3]–[5] non-existence results were obtained for coupled static systems by choosing polar coordinates and analyzing the nonlinear radial ODEs. Recently, M. Dafermos [1] proposed a different method where he analyzes the Einstein equations in null coordinates. His method has the advantage that it also applies to other Einstein-matter systems and generalizes to time-periodic solutions.

The mathematical and physical assumptions under which the above methods apply are quite different. Therefore, these methods are not equivalent, and it is rather subtle to decide which approach is preferable for a given physical system. The purpose of this short note is to compare the different approaches by collecting and discussing the necessary assumptions and the obtained results.

The physical situation of interest is the spherically symmetric collapse to a black hole. Thus thinking of the Cauchy problem for a coupled Einstein-matter system, we consider an initial Cauchy surface which is topologically $\mathbb{R}^3$, such that an event horizon forms in its future development. It is a reasonable physical assumption that asymptotically for large time, the system should settle down to a static (or more generally time-periodic) system. Under this assumption, ruling out non-trivial static solutions means that the matter (as described by the Dirac field) is no longer present asymptotically as $t \to \infty$, and thus the matter must either have fallen into the black hole or must have escaped to infinity. For this physical interpretation to hold, it is essential that the assumptions, under which the non-existence result for the static Einstein-Dirac system applies, are satisfied in the gravitational collapse.

We now discuss the individual papers in chronological order. In the first paper [2] the situation is particularly simple in that the gravitational field is a given Reissner-Nordström background field. In the non-extreme case, the problem is analyzed in the maximally extended Kruskal space-time, where the domain of outer communications D is connected to both a black hole and a white hole through the event horizons H_+ and H_-, respectively (see the conformal diagram in Figure 1; the figures are taken from [1]). In the context of the gravitational collapse described above, the Reissner-Nordström metric should be considered as physical space-time only asymptotically as $t \to \infty$. In particular, the physical metric should involve a black hole, but no white hole. In order to take this

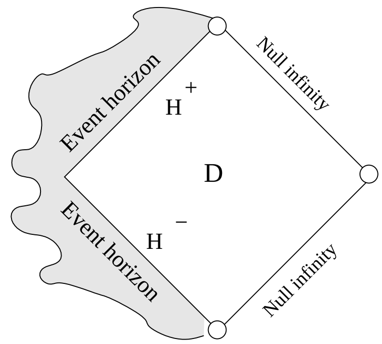

Figure 1: Conformal diagram of the extended space-time

physical input into account in the time-periodic situation, it was imposed that the integral of the Dirac current over the event horizon of the white hole H_- be zero. This assumption means that no matter should come out of the white hole, corresponding to the fact that in the physical space-time no white hole is present.

The physical picture of the gravitational collapse also tells us that one must be very careful in imposing regularity conditions for the wave function Ψ near the event horizons. First of all, we recall that due to current conservation, the probability integral

$$\int_{\mathcal{H}} \overline{\Psi} G^k \Psi \, \nu_k \, d\mu_{\mathcal{H}}$$

is independent of the choice of the hypersurface $\mathcal{H}$. Here the integrand (also called the "probability density") is positive when the hypersurface is spacelike, but it may be negative when the hypersurface is non-spacelike. This causes problems when we consider the probability integral over the hypersurface $t =$ const in Schwarzschild coordinates up to the event horizon $r = r_1$. Namely, the hypersurface $t =$ const is spacelike outside and non-spacelike inside the event horizon. The gravitational collapse tells us that the probability integral over the hypersurface $t =$ const should be finite when we integrate across the horizon. But since the integrand may be negative inside the horizon, there seems no reason why the probability integrals inside and outside the event horizons should separately be finite. Because of this difficulty, in [2] it was assumed merely that the probability integral is finite *outside and away from the event horizon*, i.e. when we integrate from $r = r_1 + \varepsilon$ to $r = \infty$. Transforming a plane wave $\sim e^{i\omega t}$ to Kruskal coordinates, one sees furthermore that functions which are smooth in Schwarzschild coordinates will in Kruskal coordinates in general be highly singular near H^+ and H^-. This regularity problem is bypassed in [2] by evaluating the Dirac equation in a neighborhood of the event horizons *only weakly.* This gives rise to so-called matching conditions. Using that the Dirac current is divergence-free, we finally show that the matching conditions are incompatible with the finite probability integral outside and away of the event horizon unless Ψ vanishes identically (= radial flux argument). In the extreme case, a completely different method involving a barrier argument for the spinors is used, but we shall not discuss this here.

In the subsequent papers [3]–[5] various static Einstein-Dirac systems are analyzed. Because of the coupling to matter, nothing is known about the asymptotic form of the metric near the event horizon. Therefore, we cannot extend the metric across the event horizon, and it is impossible to derive matching conditions. For this reason, the solutions are considered only outside the event horizon. It is convenient to choose polar coordinates

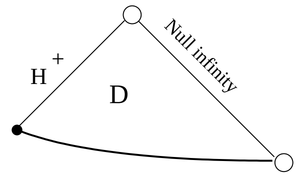

Figure 2: Conformal diagram of physical space-time

$(t, r, \vartheta, \varphi)$ where the problem reduces to analyzing the radial ODEs. As in [2], it is merely assumed that the probability integral should be zero outside and away of the event horizon. Furthermore, it is assumed that the volume element $\sqrt{|\det g_{ij}|}$ is smooth and non-zero on the event horizon; this means physically that an observer who is freely falling into the black hole should not feel strong tidal forces when crossing the horizon. Finally, a regularity assumption is needed for the metric function $A(r)$ near the event horizon. More precisely, in [3] and [5] it is assumed that A has near the event horizon a power expansion of the form $A(r) \sim (r - r_1)^\kappa + o((r - r_1)^\kappa)$ with $\kappa > 0$. In [4] only monotonicity of A near $r = r_1$ is needed. The regularity assumption on A has no physical motivation, it is a purely technical assumption. On the other hand, this assumption is very weak and seems to cover all cases of physical interest.

We finally discuss the approach by Dafermos [1]. His interesting methods make it possible to treat general classes of spherical systems that are either static or time-periodic. In contrast to [3]–[5], where the equations are considered in polar coordinates, Dafermos uses null coordinates. He also has gravitational collapse in mind and thus assumes that physical space-time has a conformal diagram as shown in Figure 2. As explained above, the physical picture is that space-time should go over to a static space-time asymptotically near time-like infinity. Dafermos needs that this static space-time has a conformal diagram with two intersecting event horizons $H^\pm$ as shown in Figure 1. This assumption is clearly satisfied in the non-extreme Reissner-Nordström metric (as considered in [2]). But for a coupled Einstein-Dirac system, there seems no reason why this assumption should be satisfied. In particular, physical arguments do not apply because physical space-time is static only asymptotically near time-like infinity, and thus it is unphysical to consider the static space-time for all (also negative) times. The assumption that static space-time has two intersecting event horizon implies that the event horizon is of the same type as in the non-extreme Reissner-Nordström geometry (i.e. that a neighborhood of the origin in the conformal diagram in Figure 1 is diffeomorphic to the neighborhood of the origin in Kruskal coordinates). This is a very strong assumption, which excludes in particular the power behavior $A(r) \sim (r - r_1)^\kappa$, $\kappa \neq 1$, as considered in [3]. Moreover, Dafermos imposes very strong regularity assumptions by assuming that the Dirac current $\overline{\Psi} G^\alpha \Psi$ is locally C^1. This implies in particular that the probability integral is finite if one integrates up to the event horizon. As discussed above, this too is a strong assumption. Finally, the assumption $\overline{\Psi} G^\alpha \Psi \in C^1_{\text{loc}}$ is also problematic because, as explained above, even smooth functions in polar coordinates will in general be very singular on $H^\pm$.

To summarize, the approach [1] has the advantage that it applies to a general class of systems and to time-periodic solutions. However, when specialized to Einstein-Dirac systems, it requires strong assumptions on the causal structure of static space-time and on the regularity of the wave function Ψ. In physical situations where these strong assumptions do not obviously hold, the methods in [2]–[5] are preferable. All methods have in common

that particular properties of the Dirac current $\overline{\Psi} G^\alpha \Psi$ are used: it is a divergence-free, future-directed time-like vector field. Thus the current conservation and the positivity of the probability density are a key ingredient to all non-existence proofs.

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